Investigating Constructions of Macaulay Posets and Rings

Penelope Beall
Joint work with Erenay Boyali, Nancy Chen, Ellen Chlachidze, Trong Toan Dao, Frederic Garvey, Mitchell Johnson, Yu Olivier Li, Nikola Kuzmanovski, Kelvin Ma, Treanungkur Mal, Rukshan Marasinghe, Quinlan Mayo, Nava Minsky-Primus, Alexandra Seceleanu, and Sriram Veerapaneni

University Math Society
January 24th, 2025

Posets

A **partially ordered set** (poset) $(\mathcal{P}, \leq)$ is a set $\mathcal{P}$ equipped with an partial order relation $\leq$ such that

- **(Reflexivity)** For all $p \in \mathcal{P}$, we have that $p \leq p$
- **(Antisymmetry)** For all $a, b \in \mathcal{P}$, if $a \leq b$ and $b \leq a$, then $a = b$
- **(Transitivity)** For all $a, b, c \in \mathcal{P}$, if $a \leq b$ and $b \leq c$, then $a \leq c$

A **totally ordered set** is a poset with the additional axiom

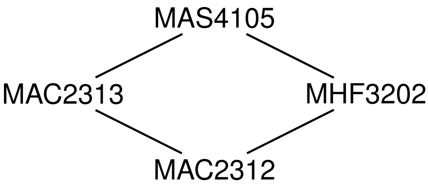
- **(Strongly connected)** For all $a, b \in \mathcal{P}$, either $a \leq b$ or $b \leq a$

Hasse diagrams

We can represent posets using **Hasse diagrams**. Let $(\mathcal{P}, \leq)$ be a poset. Then each $p \in \mathcal{P}$ is a vertex with an upward edge from p to q whenever q **covers** p, i.e., we have that

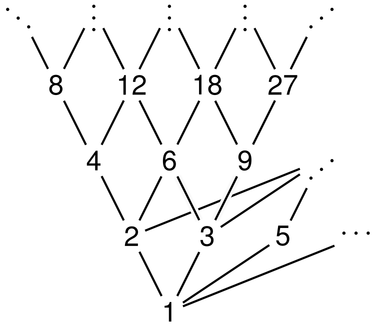
- $p < q$
- There is no r such that $p < r < q$

Example In the poset of UF courses where "$p \leq q$" means "p is a prerequisite for q," the relation MAC2312 $\leq$ MAS4105 is not a covering relation.

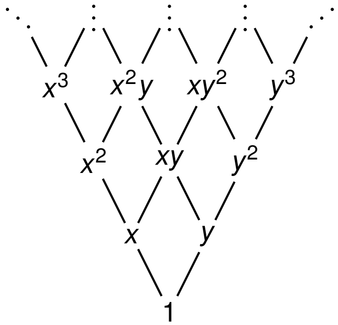


More examples of posets

Natural numbers $\mathbb{N}$ with divisibility: $a \leq b$ iff $ac = b$ for some c



Monomials in $K[x, y]$ with divisibility: $a \leq b$ iff $ac = b$ for some c



Rings 🪐

A **ring** $(R, +, *, 1)$ is a set R with binary operations $+, *$ and a multiplicative identity 1 such that

$$(R, +) \text{ is an abelian group}$$
$$(R, *) \text{ is a monoid}$$
$$* \text{ distributes over } +$$

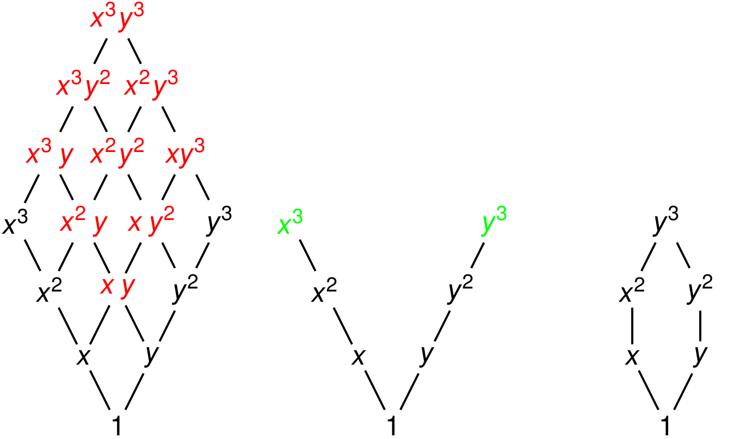
Example The set of polynomials in one variable with coefficients from a field K, denoted $R = K[x]$, is a ring.

- $(x^2 + x - 1) + (x + 1) = x^2 + 2x$
- $x(x^2 + 3) = x^3 + 3x$

Example Polynomials in several variables $K[x_1, x_2, \ldots, x_n]$

Example Quotients of polynomial rings $\dfrac{K[x_1, x_2 \ldots, x_n]}{(f_1, f_2, \ldots, f_m)}$

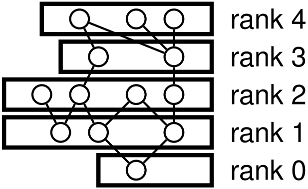
Quotients of polynomial rings



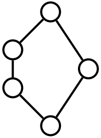
$$\frac{K[x,y]}{(x^4,y^4)} \qquad \frac{K[x,y]}{(x^4,y^4,xy)} \qquad \frac{K[x,y]}{(x^4,y^4,xy,x^3-y^3)}$$

Ranked posets

A poset $\mathcal{P}$ is **ranked** if there exists a rank function $r\colon \mathcal{P} \to \mathbb{N}$ such that $r(a) + 1 = r(b)$ whenever b covers a.



Ranked poset



Non-ranked poset

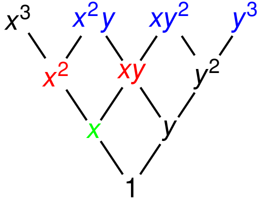
Example Monomial posets are ranked by degree, e.g. $r(x^2y) = 3$.

Shadows and segments

For a subset A of a poset $\mathcal{P}$, the **upper shadow** of A is

$$\nabla_{\mathcal{P}} A = \{p \in \mathcal{P} \mid p \text{ covers } a \text{ for some } a \in A\}\,.$$

If a poset comes with an additional total order, the **initial segment** $\operatorname{Seg}_d q$ is the largest q elements of rank d with respect that total order.



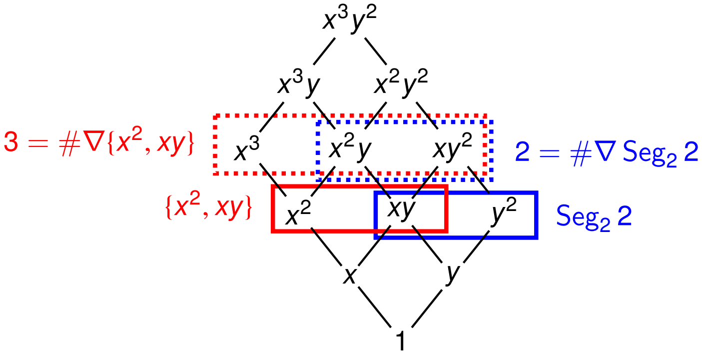
- $\nabla_{\mathcal{P}}\{x\} = \{x^2, xy\}$
- $\operatorname{Seg}_3 3 = \{x^2y, xy^2, y^3\} = \nabla_{\mathcal{P}}\{xy, y^2\} = \nabla_{\mathcal{P}} \operatorname{Seg}_2 2$

Macaulay posets

Let $(\mathcal{P}, \leq)$ be a ranked poset. Then $\mathcal{P}$ is **Macaulay** iff there exists a total order $\mathcal{O}$ such that

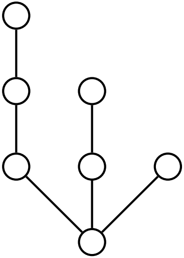
- **(Nestedness)** Initial segments have the smallest upper shadows.
- **(Continuity)** The upper shadow of an initial segment is an initial segment.

A ring is **Macaulay** iff its monomial poset is Macaulay.

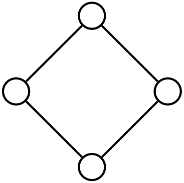


Examples of Macaulay posets

Chain

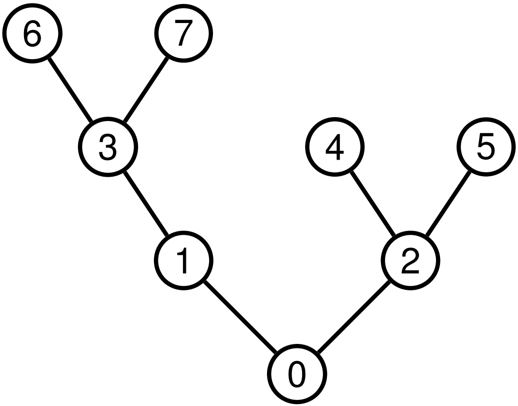


Spider



Box

Non-Macaulay poset

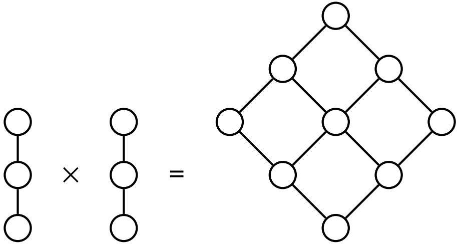


Cartesian product $\times$

Let $\mathcal{P}_i$ with $1 \leq i \leq n$ be posets. Their **cartesian product** is

$$\mathcal{P}_1 \times \mathcal{P}_2 \times \cdots \times \mathcal{P}_n = \{(p_1, \cdots, p_n) \mid p_i \in \mathcal{P}_i\}$$

where $(p_1, \cdots, p_n) \leq (p'_1, \cdots, p'_n)$ if and only if $p_i \leq p'_i$ for $1 \leq i \leq n$.
Example A **box** is a product of chains.

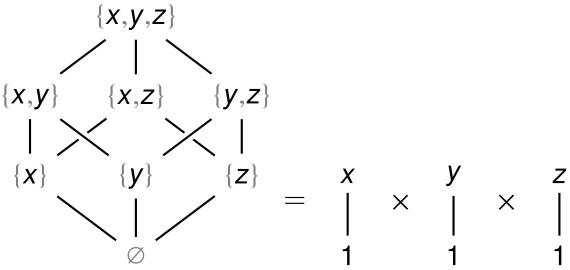


Theorem [Macaulay 1927] The ring $K[x_1, \ldots, x_n]$ is Macaulay.
Theorem [Clements-Lindström 1969] $\frac{K[x_1,\ldots,x_n]}{(x_1^{d_1},\ldots,x_n^{d_n})}$ is Macaulay.

Cartesian product ×

Example The poset of subsets of $\{x, y, z\}$ is

$$2^{\{x,y,z\}} \cong \{1, x\} \times \{1, y\} \times \{1, z\} .$$



This is also the monomial poset of

$$\frac{K[x, y, z]}{(x^2, y^2, z^2)} \cong \frac{K[x]}{(x^2)} \otimes \frac{K[y]}{(y^2)} \otimes \frac{K[z]}{(z^2)} .$$

Tensor product ⊗

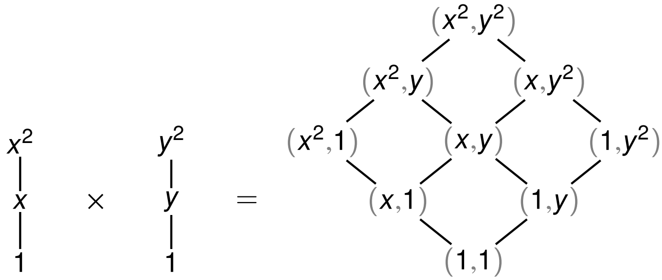
The **tensor product** of rings is the following.

$$\frac{K[x_1,\ldots,x_m]}{(f_1,\ldots f_p)} \otimes \frac{K[y_1,\ldots,y_n]}{(g_1,\ldots,g_q)} = \frac{K[x_1,\ldots,x_m,y_1,\ldots,y_n]}{(f_1,\ldots f_p,g_1,\ldots,g_q)}$$

Analogous to Cartesian product: $\mathcal{M}(S) \times \mathcal{M}(T) = \mathcal{M}(S \otimes T)$ [Kuzmanovski 2023].

Example

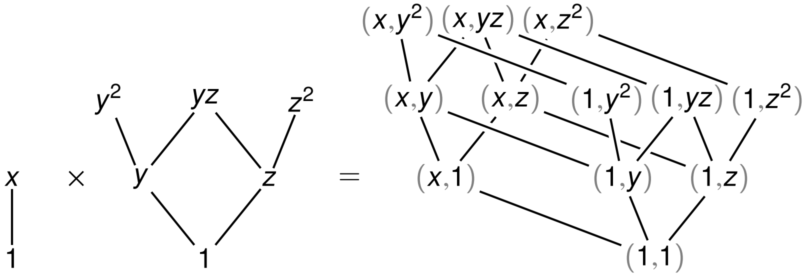
$$\frac{K[x]}{(x^3)} \otimes \frac{K[y]}{(y^3)} = \frac{K[x,y]}{(x^3,y^3)}$$



Tensor product ⊗

Example Here's a non-Macaulay tensor product of Macaulay rings.

$$\frac{K[x]}{(x^2)} \otimes \frac{K[y,z]}{(y^3, y^2z, yz^2, z^3)} = \frac{K[x,y,z]}{(x^2, y^3, y^2z, yz^2, z^3)}$$

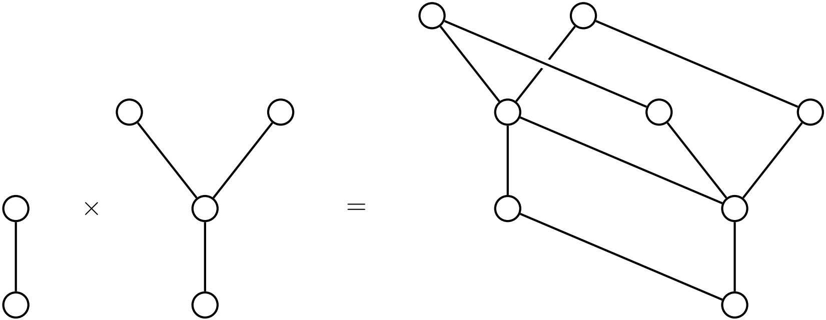


Conjecture Suppose R is a Macaulay ring whose ideal is generated by monomials. Suppose n is larger than the highest degree of a monomial in R. Then $\frac{K[x]}{(x^n)} \otimes R$ is Macaulay.

Theorem [Mermin and Peeva 2007] This is true for $n = \infty$.

Cartesian product ×

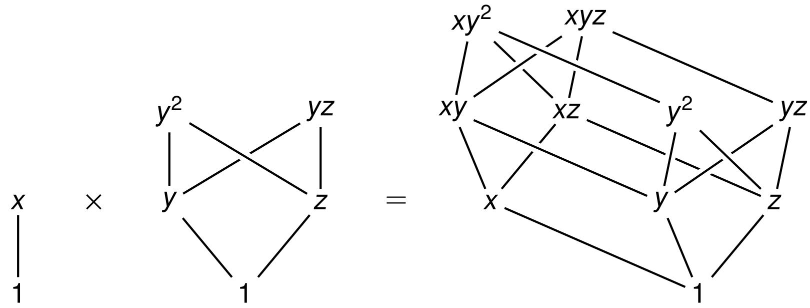
Here's the smallest non-Macaulay Cartesian product of two non-trivial Macaulay posets.



Tensor product $\otimes$

Here's the smallest non-Macaulay Cartesian product of two non-trivial Macaulay monomial posets of rings.

$$\frac{K[x]}{(x^2)} \otimes \frac{K[y,z]}{(y^3, y^2 - z^2, z^3)} = \frac{K[x,y,z]}{(x^2, y^2 - z^2, y^3, z^3)}$$



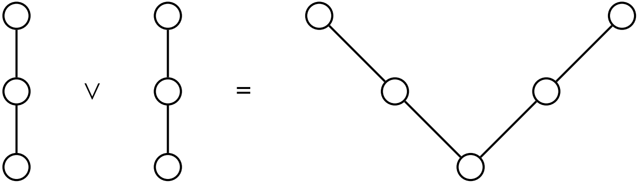
Wedge Product $\vee$

Suppose each $(\mathcal{P}_i, \leq_i)$ has a unique minimum element ℓ_i. The **wedge product** of the $\mathcal{P}_i$ is

$$\mathcal{P}_1 \vee \mathcal{P}_2 \vee \cdots \vee \mathcal{P}_n = \left(\bigsqcup_{i=1}^{n} \mathcal{P}_i \right) / (\ell_1 = \ell_2 = \cdots = \ell_n)$$

where $a \leq b$ if and only if $a \leq_i b$ in $\mathcal{P}_i$ for some i.

Example A **spider** is a wedge of chains.



Theorem [Bezrukov–Elässer 2000] Spiders are Macaulay.

Fiber product over K

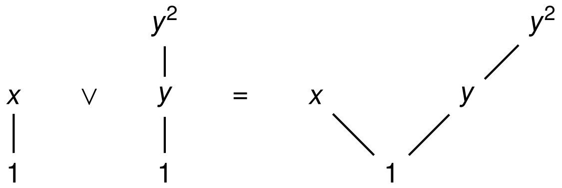
The **fiber product over** K of rings is the following.

$$\frac{K[x_1,\ldots,x_m]}{I}\times_K\frac{K[y_1,\ldots,y_n]}{J}=\frac{K[x_1,\ldots,x_m,y_1,\ldots,y_n]}{I+J+(x_iy_j\mid 1\le i\le m,\,1\le j\le n)}$$

This is analogous to wedge product: $\mathcal{M}(S)\vee\mathcal{M}(T)=\mathcal{M}(S\times_K T)$.

Example

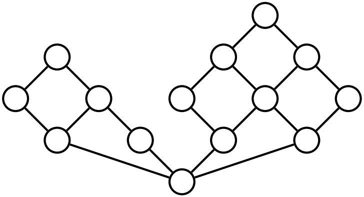
$$\frac{K[x]}{(x^2)}\times_K\frac{K[y]}{(y^3)}=\frac{K[x,y]}{(x^2,y^3,xy)}$$



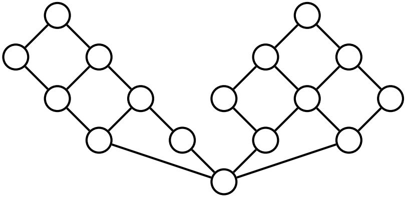
Wedge product of boxes $\square \vee \square = \square \times_K \square$

Theorem If $\mathcal{P}$ and $\mathcal{Q}$ are box posets of shape $m \times n$ and $m' \times n'$, respectively, such that $2 \leq m \leq n$ and $2 \leq m' \leq n'$, then their wedge $\mathcal{P} \vee \mathcal{Q}$ is Macaulay if and only if $m \leq m'$ and $n \leq n'$ or vice versa.

Example



2×3 wedge 3×3
Macaulay



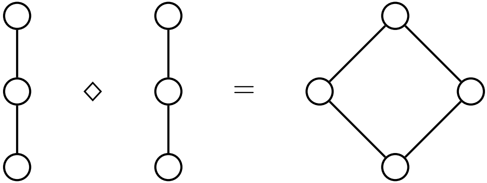
2×4 wedge 3×3
Not Macaulay

Diamond product $\diamond$

Suppose ℓ_i, L_i are respectively the unique minimal and maximal element of $(\mathcal{P}_i, \leq_i)$. The **diamond product** is

$$\mathcal{P}_1 \diamond \mathcal{P}_2 \diamond \cdots \diamond \mathcal{P}_n = \left(\bigsqcup_{i=1}^{n} \mathcal{P}_i \right) / (\ell_1 = \ell_2 = \cdots = \ell_n, L_1 = L_2 = \cdots = L_n)$$

where $a \leq b$ if and only if $a \leq_i b$ in $\mathcal{P}_i$ for some i.



Connected sum $\#_K$

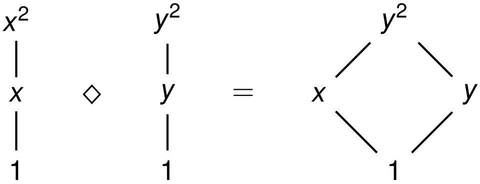
The **connected sum over** K of rings is the following.

$$S \#_K T = \frac{S \times_K T}{\max \mathcal{M}(S) - \max \mathcal{M}(T)}$$

Analogous to diamond product: $\mathcal{M}(S) \diamond \mathcal{M}(T) = \mathcal{M}(S \#_K T)$

Example

$$\frac{K[x]}{(x^3)} \#_K \frac{K[y]}{(y^3)} = \frac{K[x,y]}{(x^3, y^3, xy, x^2 - y^2)}$$



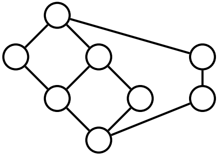
Diamond product of boxes $\square \diamond \square$

Theorem

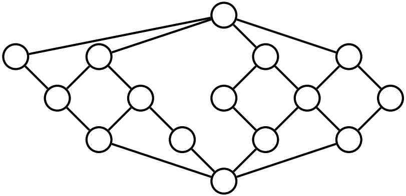
If $\mathcal{P}$ and $\mathcal{Q}$ are box posets of the same rank, then $\mathcal{P} \diamond \mathcal{Q}$ is Macaulay if and only if one of the following is true:

- $\mathcal{P}$ is the same as $\mathcal{Q}$ or
- $\mathcal{P} =$ path poset, $\mathcal{Q} = 2 \times q$ box poset or vice-versa.

Example



2×3 diamond 1×4
Macaulay



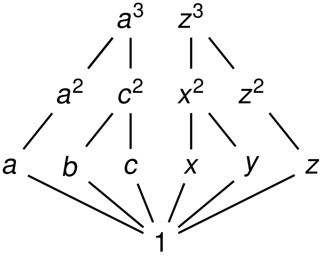
2×4 diamond 3×3
Not Macaulay

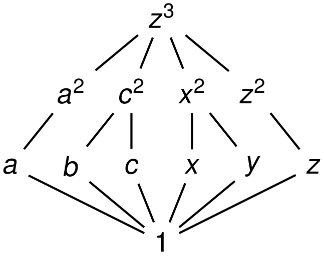
Relations among $\sqcup$, $\vee$, $\diamond$

Theorem

$$\bigsqcup_{i=1}^{n} \mathcal{P}_i \text{ is Macaulay} \implies \bigvee_{i=1}^{n} \mathcal{P}_i \text{ is Macaulay} \implies \mathop{\diamond}_{i=1}^{n} \mathcal{P}_i \text{ is Macaulay}$$

Example $\frac{K[x,y,z]}{(xz,yz,x^2-xy,x^2-y^2,x^3-y^3)}$ shows that $\vee \not\Leftarrow \diamond$





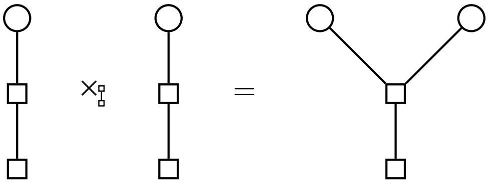
Relations among $\sqcup$, $\vee$, $\diamond$

Theorem With $r, s, t > 1$, the following are equivalent:

- $\mathcal{P}$ is *additive*
- $\bigsqcup_{i=1}^{r} \mathcal{P}$ is Macaulay wrt the *union simplicial order*
- $\bigvee_{i=1}^{s} \mathcal{P}$ is Macaulay wrt the *union simplicial order*
- $\Diamond_{i=1}^{t} \mathcal{P}$ is Macaulay wrt the *union simplicial order*

Fiber Product

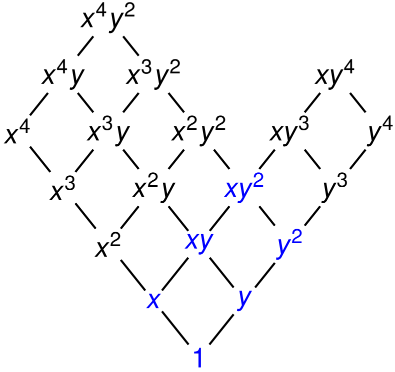
If posets $\mathcal{P}$ and $\mathcal{Q}$ have a common subposet $\mathcal{R}$, the **fiber product** $\mathcal{P} \times_{\mathcal{R}} \mathcal{Q}$ takes the disjoint union and identifies corresponding elements in $\mathcal{R}$.



Heart posets $\square \times_{\square} \square =$ 💙

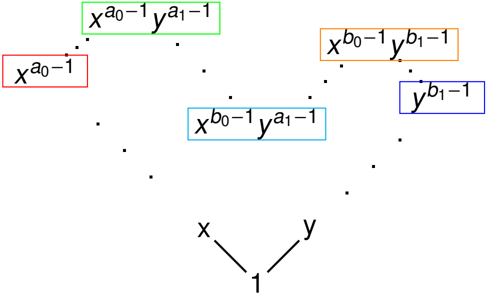
A **heart poset** is obtained by taking the fiber product of two box posets along their common box subposet.

Example A 5×3 box glued to a 2×5 box along their common 2×3 subbox.



Macaulay heart posets

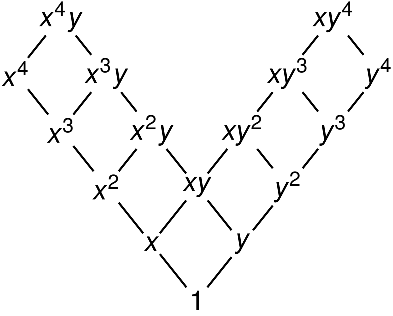
Theorem Assuming the red node has rank higher than the blue node, the heart poset is Macaulay if and only if the green node has rank higher than the orange node.



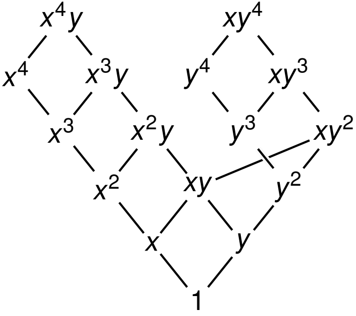
The order with respect to which the poset is Macaulay depends on whether the cyan node is above or below the blue node.

Orders in heart posets

If the cyan node has rank above a certain threshold, we use what is called the lex order. Else, we use the twist order.



Lex order



Twist order

Applications

Macaulay posets and rings are used in **enumerative combinatorics** and **commutative algebra**. I'm not personally familiar with these applications, but here's part of the introduction section of Nikola Kuzmanovski's paper:

The Macaulay property on posets provides several applications in pure mathematics and engineering. If a poset is Macaulay with a rank-greedy order then a solution to the maximum weight ideal problem follows. Edge-isoperimetric problems on graphs reduce to the maximum weight ideal problem, whence the Macaulay property gives solutions to edge-isoperimetric problems as well. These implications were observed by Bezrukov in [6], and have been included among other discrete extremal problems in [25, 31]. Edge-isoperimetric problems give solutions to many other problems. The survey [5] and the book [31] go over some, including the wirelength problem, the bisection width and edge congestion problem, and graph partitioning problems. It is interesting to note that solving the edge-isoperimetric problem on the Petersen graph [7] was motivated by application to parallel processing. For an application in pure mathematics, Daykin [24] showed that the Erdös-Ko-Rado Theorem follows from the Kruskal-Katona Theorem.

The study of Macaulay rings (Definition 2.5.6) started almost a century ago with Macaulay [46]. Macaulay's Theorem says that for every homogeneous ideal in a polynomial ring over a field, there exists a monomial lex ideal with the same Hilbert function. There has been a lot of interest in generalizing Macaulay's Theorem to quotients of polynomial rings [28, 29, 47, 48, 49, 50, 55, 56]. Lex ideals play an important role in Hartshorne's proof that the Hilbert scheme is connected [32]. Another very remarkable result is due to Bigatti [13], Hulett [34] and Pardue [51], which states that lex ideals in a polynomial ring have the largest graded Betti numbers among all ideals with the same Hilbert function.

References

P. Beall, E. Boyali, N. Chen, E. Chlachidze, T.T. Dao, F. Garvey, M. Johnson, Y.O. Li, N. Kuzmanovski, K. Ma, T. Mal, R. Marasinghe, Q. Mayo, N. Minsky-Primus, A. Seceleanu, S. Veerapaneni, *New Classes of Macaulay Posets and Macaulay Rings*, in preparation.

P. Beall, N. Kuzmanovski, Y. O. Li and A. Seceleanu, *The MacaulayPosets package for Macaulay2*, in preparation.

S. L. Bezrukov and R. Elsässer, *The spider poset is Macaulay*, Journal of combinatorial theory, Series A 90, 1 (2000), 1–26.

G. Clements and B. Lindström, *A generalization of a combinatorial theorem of Macaualay*, Journal of combinatorial theory 7, 3 (1969), 230–238.

D. Grayson and M. Stillman, *Macaulay2, a software system for research in algebraic geometry*, available at https://faculty.math.illinois.edu/Macaulay2/.

A. Iarrobino, C. McDaniel,A. Seceleanu, *Connected Sums of Graded Artinian Gorenstein Algebras and Lefschetz Properties*, J. Pure Appl. Algebra 226 (2022), no. 1, 106787.

N. Kuzmanovski, *Macaulay Posets and Rings*, preprint (2023), available at https://arxiv.org/pdf/2307.05094.

F. S. Macaulay, (1927). Some Properties of Enumeration in the Theory of Modular Systems. Proceedings of the London Mathematical Society, s2-26(1), 531555. https://doi.org/10.1112/plms/s2-26.1.531.

J. Mermin and I. Peeva, *Lexifying ideals*, Math. Res. Lett.13 (2006), no.2-3, 409–422.

Acknowledgements

We thank Dr. Alexandra Seceleanu and Dr. Nikola Kuzmanovski for mentoring us for the Polymath Jr. 2024 REU, funded by NSF grant DMS-2341670.

Macaulay2

The MacaulayPosets package was developed for the computer algebra system Macaulay2. An example session is shown below

```
i1 : loadPackage ‘‘MacaulayPosets’’
i2 : S = QQ[x, y, z]/(x^4, y^2-z^2, z^2-x*y)
i3 : isMacaulay S
o3 : true
i4 : isMacaulay ringConnectedSum(S, S)
o4 : false
```