Schubert polynomials as generating polynomials

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University of Florida

University Math Society
Spring 2025 Undergraduate Talk Extravaganza
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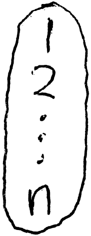
Permutations

Let $[n]$ denote the n-element set $\{1, 2, \ldots, n\}$.



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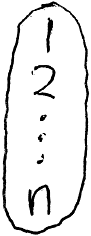
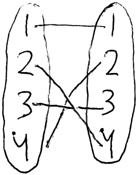


Then $[\infty] = \{1, 2, \ldots\} = \mathbb{Z}^{+}$.

Permutations

A **permutation** is a bijection $[n] \to [n]$, i.e. a pairing of elements in one set with elements in another set.



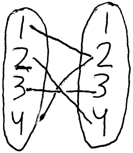
The permutation 1432

$1 \mapsto 1$

$2 \mapsto 4$

$3 \mapsto 3$

$4 \mapsto 1$



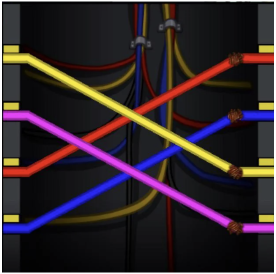
Not a permutation

$1 \mapsto 2 \leftarrow\!\shortmid 4$

A permutation σ can be written $\sigma(1)\sigma(2)\cdots\sigma(n)$, like 1432 above

Permutations

Example. The Fix Wires task in Among Us involves permuting wires correctly



The permutation 3412

$$1 \mapsto 3$$
$$2 \mapsto 4$$
$$3 \mapsto 1$$
$$4 \mapsto 2$$

Groups

A **group** is a collection of elements along with a nice way of combining any pair of elements into a new element.

Example. Integers with addition

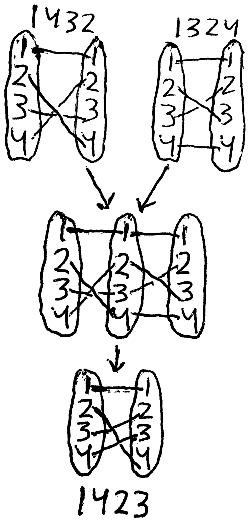
$$(2+2)-2=2+(2-2)=2+0=2$$

Example. Symmetries of a molecule

Example. The symmetric group S_n

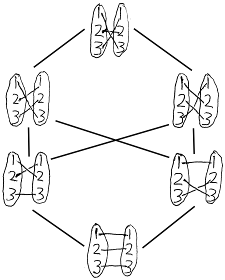
Groups

Permutations live in the **symmetric group**



Posets

Permutations also live in a poset



This provides a recursive way to define **Schubert polynomials**: with $\mathfrak{S}_{\mathrm{id}} = 1$, if $w \in S_n$ and (r, s) is the lex largest inversion of w and $v = wt_{rs}$, then

$$\mathfrak{S}_w = x_r \mathfrak{S}_v + \sum_{\substack{q<r \\ v \lessdot vt_{qr}}} \mathfrak{S}_{vt_{qr}} \, .$$

Posets

Repeatedly applying the Transition Equation

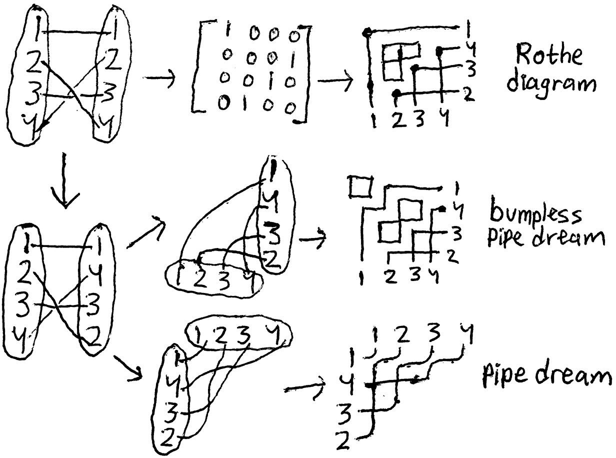
$$\mathfrak{S}_w = x_r \mathfrak{S}_v + \sum_{\substack{q<r \\ v \lessdot vt_{qr}}} \mathfrak{S}_{vt_{qr}} \,,$$

the Schubert polynomial for 1432 is

$$\begin{aligned} \mathfrak{S}_{1432} &= x_3 \mathfrak{S}_{1423} + \mathfrak{S}_{2413} \\ &= x_3 (x_2 \mathfrak{S}_{1324} + \mathfrak{S}_{3124}) + x_2 \mathfrak{S}_{2314} + \mathfrak{S}_{3214} \\ &= x_3 (x_2 (x_2 + \mathfrak{S}_{2134}) + x_1 \mathfrak{S}_{2134}) + x_2^2 \mathfrak{S}_{2134} + x_2 \mathfrak{S}_{3124} \\ &= x_3 (x_2^2 + x_1 x_2 + x_1^2) + x_2^2 x_1 + x_2 x_1 \mathfrak{S}_{1324} \\ &= x_2^2 x_3 + x_1 x_2 x_3 + x_1^2 x_3 + x_1 x_2^2 + x_1 x_2^2 \,. \end{aligned}$$

Permutations

There are many ways to represent a permutation



Pipe dream

A **pipe dream** is a finite subset D of $\mathbb{Z}^+ \times \mathbb{Z}^+$.

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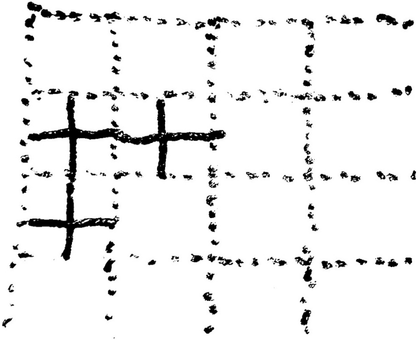
1. Place a cross tile at each element of D, say $D = \{(2,1),(2,2),(3,1)\}$.



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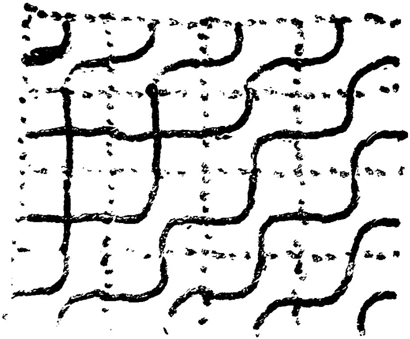
1. Place a cross tile at each element of D, say $D = \{(2,1),(2,2),(3,1)\}$.
2. Place a bump tile everywhere else



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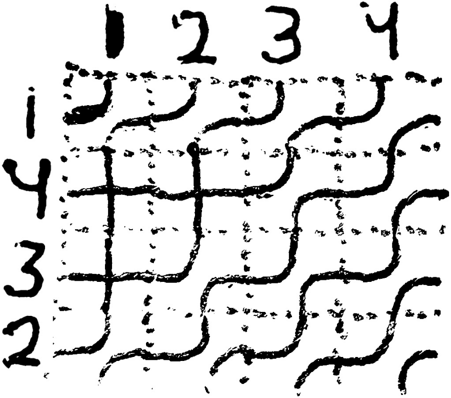
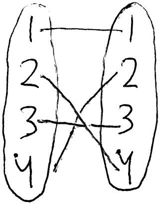
1. Place a cross tile  at each element of D, say $D = \{(2,1),(2,2),(3,1)\}$.
2. Place a bump tile  everywhere else
3. Follow the pipes from the top to see what permutation it represents



Pipe dream

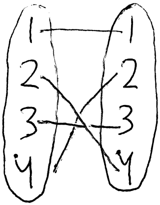
Def. An **inversion** of a permutation σ is a pair $i < j$ such that $\sigma(i) > \sigma(j)$.

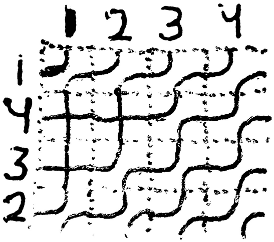
Example. $\text{inv}(1432) = \{(2,3),(2,4),(3,4)\}$, wherever the wires cross



Pipe dream

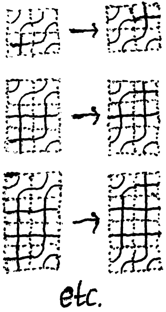
Def. An **inversion** of a permutation σ is a pair $i < j$ such that $\sigma(i) > \sigma(j)$.
Def. If c_i denotes the number of inversions of σ starting with i, the **bottom pipe dream** of σ is has c_i cross tiles left-justified in row i.
Example. $\text{inv}(1432) = \{(2,3),(2,4),(3,4)\}$, wherever the wires cross





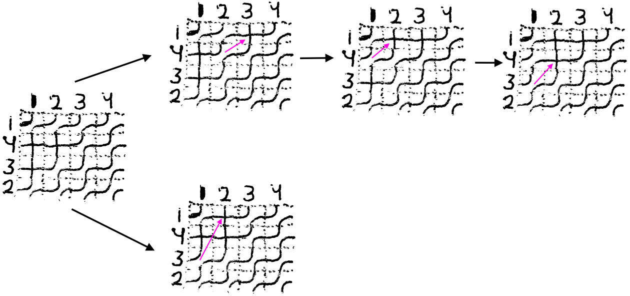
Pipe dream

Applying a **ladder move** to a pipe dream yields another pipe dream representing the same permutation



Pipe dream

Starting with the bottom pipe dream and applying all possible ladder moves yields all representatives of the permutation:



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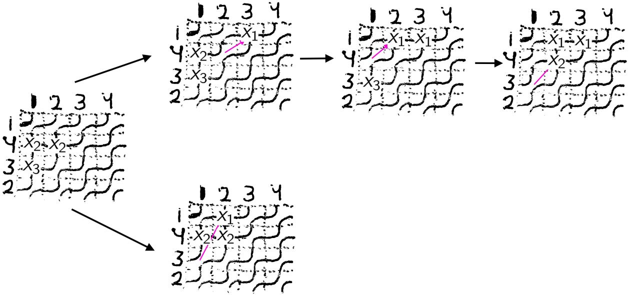

monomial weights

$x_2^2 x_3 \quad x_1 x_2^2 \quad x_1 x_2 x_3 \quad x_1^2 x_3 \quad x_1^2 x_2$

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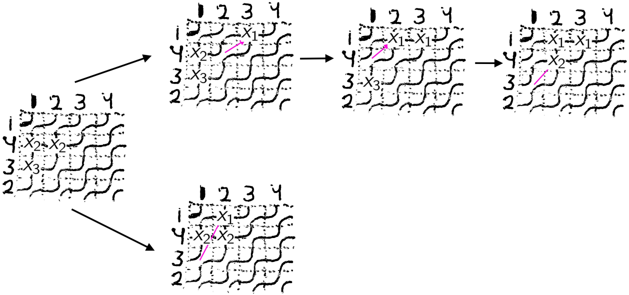
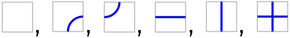


Summing all the monomial weights yields a Schubert polynomial!

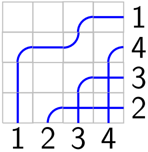
$$x_2^2x_3 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 = \mathfrak{S}_{1432}$$

Bumpless pipe dream

A **bumpless pipe dream** is a tiling of $\mathbb{Z}^+ \times \mathbb{Z}^+$ with



such that the pipes make sense.



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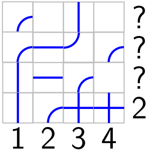

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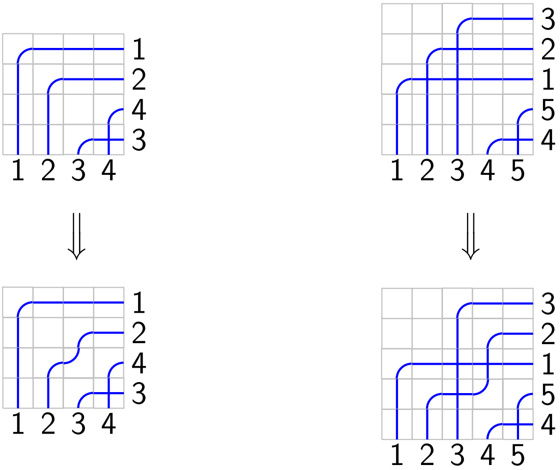
The bump tile  is now not used.

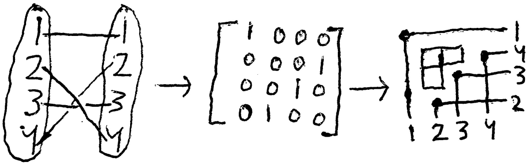
Bumpless pipe dream

Applying a **droop move** yields a new bumpless pipe dream representing the same permutation.



Bumpless pipe dream

The **Rothe diagram** of a permutation is the bumpless pipe dream where every pipe just goes up, turns right once, then goes right.



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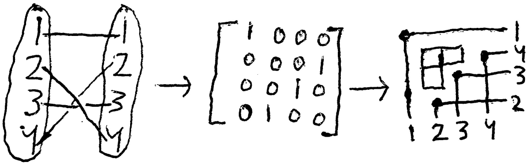

It can also be obtained from the matrix representation by replacing each 1 with an .

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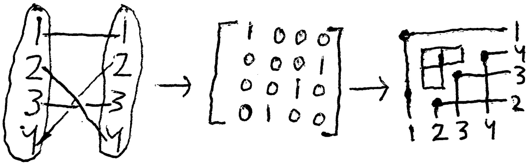
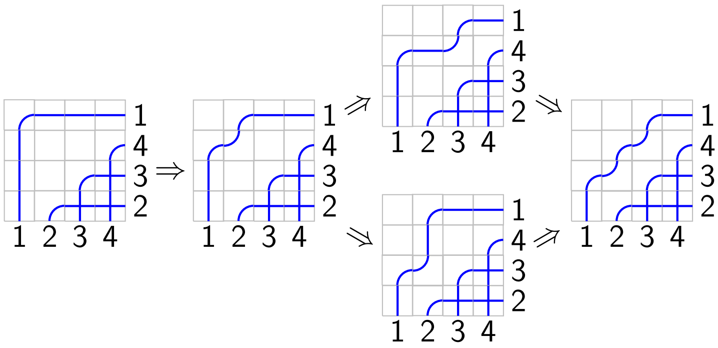


It can also be obtained from the matrix representation by replacing each 1 with an .
It can also be viewed as the set of blank tiles, which is

$$\{(i, \sigma(j)) \mid (i, j) \in \mathrm{inv}(\sigma)\}\,.$$

Bumpless pipe dream

Starting with the Rothe diagram and applying all possible droop moves yields all representatives of the permutation:



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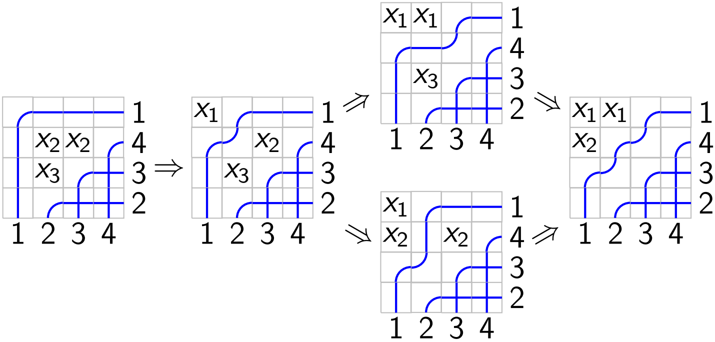

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Kohnert moves

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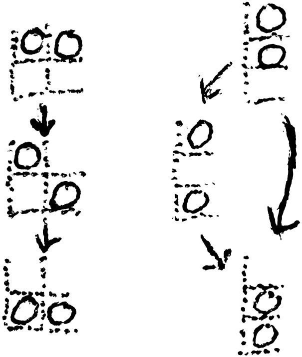
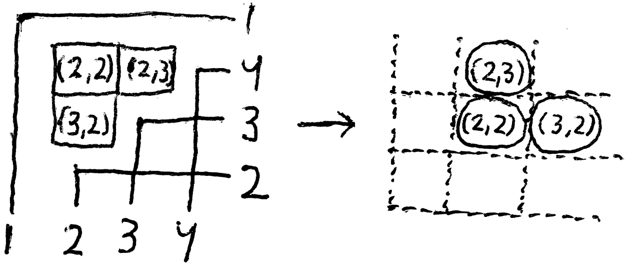
Applying a **Kohnert move** to a diagram yields a new diagram: take a rightmost cell of a row and move it downwards to the next empty spot, jumping over other cells if necessary.



Kohnert moves

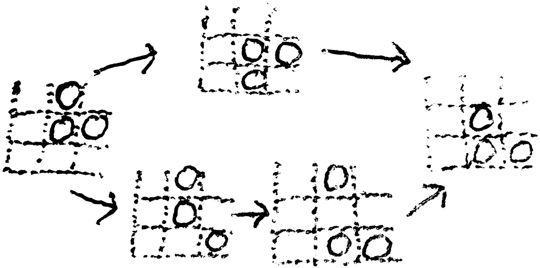
Recall that a Rothe diagram can be equivalently viewed as its set of blank tiles: $\{(i, \sigma(j)) \mid (i, j) \in \mathrm{inv}(\sigma)\} \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$



Hence, Kohnert moves are applicable to Rothe diagrams.

Kohnert moves

Starting with a Rothe diagram and applying all possible Kohnert moves yields several diagrams:



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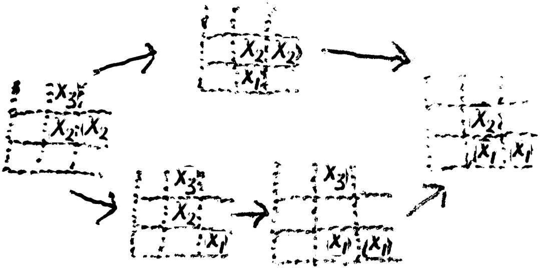

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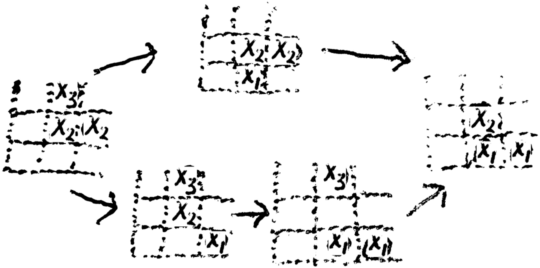

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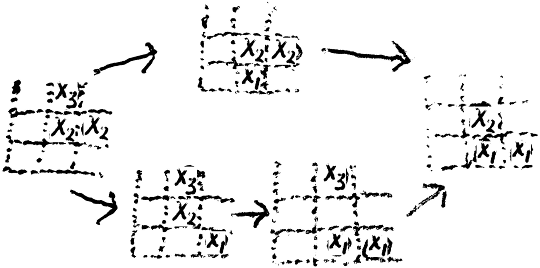

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Starting with a not-necessarily-Rothe diagram would produce a **Kohnert polynomial**, which Schubert polynomials are therefore a special case of.

References

[1] S. Assaf and D. Searles, “Kohnert Polynomials”, *Experimental Mathematics* (2022), available at https://arxiv.org/abs/1711.09498

[2] S. Billey, Y. Gao, and B. Pawlowski, “Introduction to the Cohomology of the Flag Variety,” to appear

[3] Z. Hamaker, “Combinatorics of Schubert Polynomials,” available at https://people.clas.ufl.edu/zhamaker/files/SchubertClass.pdf